Lecture 3

Some stuff I missed:

i) You can attend ANY instructor's or ANY TA's office hours.

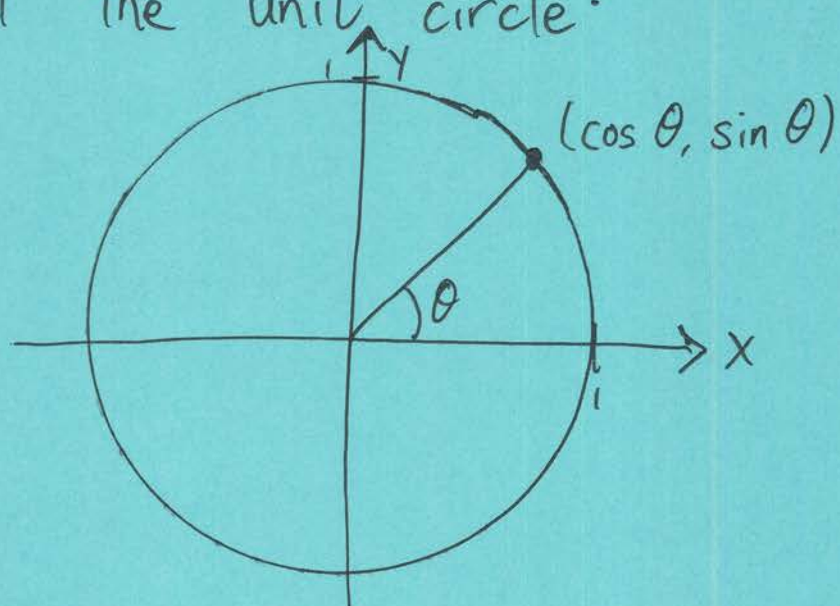
ii) Taylor's webpage (nd.edu/~taylor/Math20550) contains the course schedule

iii) Exam conflicts / missing exams

iv) Tutoring in math library

v) Course specific email: eburkard+Teaching@nd.edu

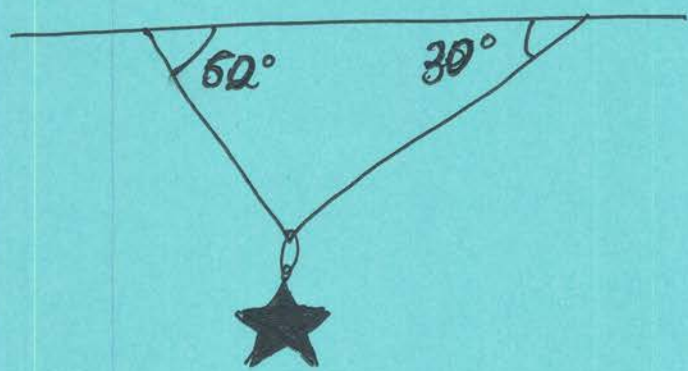
12.2 Recall the unit circle:



Using this, we can write a vector in the plane in component form. A vector, $\vec{v}$, has magnitude $|\vec{v}|$ and direction θ. In component form it looks like $\vec{v} = |\vec{v}|\langle \cos\theta, \sin\theta \rangle$

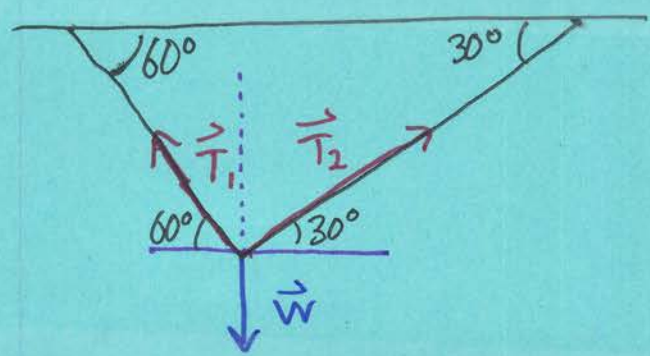
Application example

A decoration is hung on a wire as shown:



If the star weighs 15 N, find the tension in each wire.

Solution:



The tension, a vector, in each wire is $\vec{T}_1$ and $\vec{T}_2$. The weight, $\vec{w}$, is a vector pointing directly down, and its magnitude is $|\vec{w}| = 15$ N. By Newton's Third Law, we have $\vec{T}_1 + \vec{T}_2 + \vec{w} = \vec{0}$. The component form of $\vec{w}$ is easy to get, since it points in the negative y-direction:

$\vec{w} = \langle 0, -15 \rangle$

We know $\vec{T}_1 = |\vec{T}_1| \langle -\cos 60^\circ, \sin 60^\circ \rangle$

& $\vec{T}_2 = |\vec{T}_2| \langle \cos 30^\circ, \sin 30^\circ \rangle$,

so since $\vec{T}_1 + \vec{T}_2 + \vec{w} = \vec{0}$, we get the system:

$$\begin{cases} -|\vec{T}_1| \cos 60^\circ + |\vec{T}_2| \cos 30^\circ + 0 = 0 \\ |\vec{T}_1| \sin 60^\circ + |\vec{T}_2| \sin 30^\circ - 15 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -\frac{1}{2}|\vec{T}_1| + \frac{\sqrt{3}}{2}|\vec{T}_2| = 0 & (1) \\ \frac{\sqrt{3}}{2}|\vec{T}_1| + \frac{1}{2}|\vec{T}_2| = 15 & (2) \end{cases}$$

(1) $\Rightarrow |\vec{T}_1| = \sqrt{3}\,|\vec{T}_2|$. Plug this into (2):

$$\frac{\sqrt{3}}{2}\left(\sqrt{3}\,|\vec{T}_2|\right) + \frac{1}{2}|\vec{T}_2| = \frac{3}{2}|\vec{T}_2| + \frac{1}{2}|\vec{T}_2| = 2|\vec{T}_2| = 15$$

$$\Rightarrow |\vec{T}_2| = \frac{15}{2} \Rightarrow |\vec{T}_1| = \sqrt{3}\left(\frac{15}{2}\right) = \frac{15\sqrt{3}}{2}$$

$$\Rightarrow \begin{cases} \vec{T}_1 = \frac{15\sqrt{3}}{2}\langle -\cos 60^\circ, \sin 60^\circ \rangle = \left\langle -\frac{15\sqrt{3}}{4}, \frac{45}{4} \right\rangle \\ \vec{T}_2 = \frac{15}{2}\langle \cos 30^\circ, \sin 30^\circ \rangle = \left\langle \frac{15\sqrt{3}}{4}, \frac{15}{4} \right\rangle \end{cases}$$

$\diamond$

1.3 - The Dot Product

Def: The dot produt of two vectors, $\vec{u}$ and $\vec{v}$, is

- in $\mathbb{R}^2$: $\vec{u}\cdot\vec{v} = \langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle = u_1v_1 + u_2v_2$
- in $\mathbb{R}^3$: $\vec{u}\cdot\vec{v} = \langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$

Note: The dot product is a product of two vectors which outputs a scalar. The dot product is sometimes called the scalar product or the inner product.

Properties of the dot product

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors of the same dimension, and let c be a scalar.

1) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
3) $c(\vec{u}\cdot\vec{v}) = (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v})$
4) $\vec{0}\cdot\vec{v} = \vec{v}\cdot\vec{0} = \vec{0}$
5) $\vec{v}\cdot\vec{v} = |\vec{v}|^2$.

Proof of #5: (in $\mathbb{R}^3$) $\vec{v}\cdot\vec{v} = v_1v_1 + v_2v_2 + v_3v_3 = \left(\sqrt{v_1^2+v_2^2+v_3^2}\right)^2 = |\vec{v}|^2$

Ex: Given $\vec{u} = \langle 2,-2 \rangle$, $\vec{v} = \langle 5,8 \rangle$, and $\vec{w} = \langle -4,3 \rangle$,

compute: [redacted] a) $\vec{u}\cdot\vec{w}$, b) $(\vec{u}\cdot\vec{v})\vec{w}$, c) $\vec{u}\cdot(2\vec{v})$, d) $|\vec{w}|^2$

Sol: a) $\vec{u}\cdot\vec{w} = (2)(-4) + (-2)(3) = -8-6 = -14$

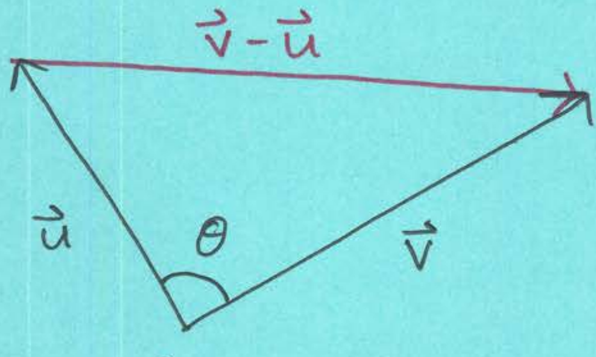
b) First $\vec{u}\cdot\vec{v} = (2)(5) + (-2)(8) = 10-16 = -6$

$(\vec{u}\cdot\vec{v})\vec{w} = -6\langle -4,3 \rangle = \langle 24,-18 \rangle$

c) $\vec{u}\cdot(2\vec{v}) = 2(\vec{u}\cdot\vec{v}) = 2(-6) = -12$

d) $|\vec{w}|^2 = \vec{w}\cdot\vec{w} = (-4)(-4) + (3)(3) = 16+9 = 25$

Consider this:



By the law of cosines (remember that?):

$$|\vec{v}-\vec{u}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$

Rewrite this using the dot product:

$$(\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2|\vec{u}||\vec{v}|\cos\theta$$

$$\Rightarrow \cancel{\vec{v}\cdot\vec{v}} - 2\vec{u}\cdot\vec{v} + \cancel{\vec{u}\cdot\vec{u}} = \cancel{\vec{u}\cdot\vec{u}} + \cancel{\vec{v}\cdot\vec{v}} - 2|\vec{u}||\vec{v}|\cos\theta$$

$\Rightarrow -2\vec{u}\cdot\vec{v} = -2|\vec{u}||\vec{v}|\cos\theta$

$\Rightarrow \boxed{\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta}$ Alternative form of dot product

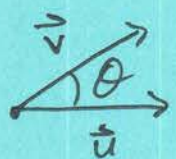
This is useful for finding angles ~~bet~~ between vectors

$\boxed{\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}} \Rightarrow \theta = \arccos\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$

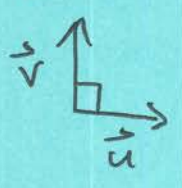
angle between $\vec{u}$ and $\vec{v}$

Using the alternative form of the dot product, we can observe how θ affects the sign of $\vec{u}\cdot\vec{v}$: $(\vec{u},\vec{v}\neq\vec{0})$

1) θ: acute $(0\leq\theta<\frac{\pi}{2}) \Rightarrow \cos\theta>0 \Rightarrow \vec{u}\cdot\vec{v}>0$



2) θ: right angle $(\theta=\frac{\pi}{2}) \Rightarrow \cos\theta \Rightarrow \vec{u}\cdot\vec{v}=0$



3) θ: obtuse $(\frac{\pi}{2}<\theta\leq\pi) \Rightarrow \cos\theta \Rightarrow \vec{u}\cdot\vec{v}<0$

<u>Ex</u>: Find the angle between

a) $\vec{u} = \langle 3,-1,2\rangle$ & $\vec{v} = \langle 2,0,1\rangle$

b) $\vec{u} = \langle 3,-1,2\rangle$ & $\vec{w} = \langle 1,-1,-2\rangle$

<u>Sol</u>: a) $\vec{u}\cdot\vec{v} = (3)(2)+(-1)(0)+(2)(1) = 8$

$|\vec{u}| = \sqrt{3^2+(-1)^2+2^2} = \sqrt{9+1+4} = \sqrt{14}$

~~$|\vec{v}| = \sqrt{2,0,1}$~~ $|\vec{v}| = \sqrt{2^2+0^2+1^2} = \sqrt{5}$

$$\Rightarrow \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{8}{\sqrt{14}\sqrt{5}} = \frac{8}{\sqrt{70}}$$

$\Rightarrow \theta = \arccos\left(\frac{8}{\sqrt{70}}\right)$ is the angle between $\vec{u}$ & $\vec{v}$

b) $\vec{u}\cdot\vec{w} = (3)(1)+(-1)(-1)+(2)(-2) = 3+1-4 = 0$

$$\Rightarrow \cos\theta = \frac{\vec{u}\cdot\vec{w}}{|\vec{u}||\vec{w}|} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

So, $\vec{u}$ & $\vec{w}$ are <u>orthogonal</u> or <u>perpendicular</u>.

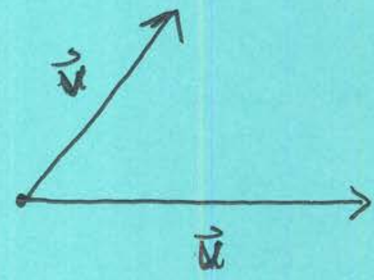
<u>General Rule</u>:

To check if two nonzero vectors $\vec{u}$ & $\vec{v}$ are orthogonal, take their dot product.

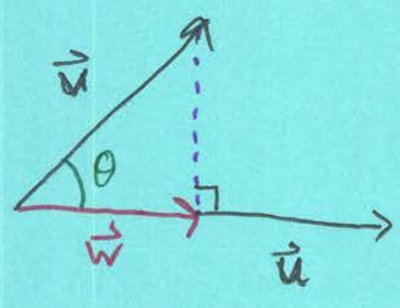
$\boxed{\vec{u} \text{ is orthogonal to } \vec{v} \text{ if } \vec{u}\cdot\vec{v} = \vec{0}}$

Projections

Given two vectors $\vec{u}$ & $\vec{v}$:



we would like to know "how much $\vec{v}$ points in the direction of $\vec{u}$," e.g, this is akin to writing the vector $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ (so $\vec{u}$ would "point in the $\hat{i}$-direction amount a"). What we are asking for is the vector $\vec{w}$ in the diagram:



$\vec{w}$ is called the <u>vector projection</u>, or simply the <u>projection</u>, of $\vec{v}$ onto $\vec{u}$, and is denoted $\text{proj}_{\vec{u}}\vec{v}$.
The signed length of this vector is called the <u>scalar projection</u> of $\vec{v}$ onto $\vec{u}$, or sometimes the the <u>component of $\vec{v}$ along $\vec{u}$</u>.
This quantity (a <u>scalar</u>!) is denoted $\text{comp}_{\vec{u}}\vec{v}$.
Observing the picture, we see $\text{comp}_{\vec{u}}\vec{v} = |\vec{v}|\cos\theta$

Recall $\vec{u}\cdot\vec{v} = |\vec{u}|\left(|\vec{v}|\cos\theta\right) \Rightarrow \text{comp}_{\vec{u}}\vec{v} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|}$

And since $\text{proj}_{\vec{u}}\vec{v}$ points in the direction of $\vec{u}$, to get $\text{proj}_{\vec{u}}\vec{v}$ we multiply the unit vector in the direction of $\vec{u}$ by $\text{comp}_{\vec{u}}\vec{v}$, i.e.,

$$\text{proj}_{\vec{u}}\vec{v} = \left(\text{comp}_{\vec{u}}\vec{v}\right)\frac{\vec{u}}{|\vec{u}|} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}\right)\frac{\vec{u}}{|\vec{u}|} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\right)\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

Ex: Find the vector and scalar projection of $\vec{v} = \langle 3,-5,2\rangle$ onto $\vec{u} = \langle 7,1,-2\rangle$

Sol: $\vec{u}\cdot\vec{v} = (3)(7)+(-5)(1)+(2)(-2) = 21-5-4 = 12$

$\cancel{|\vec{v}| = \sqrt{3^2+(-5)^2+2^2} = \sqrt{9+25+4} = \sqrt{38}}$

$|\vec{u}| = \sqrt{7^2+1^2+(-2)^2} = \sqrt{49+1+4} = \sqrt{54} = 3\sqrt{6}$

$\Rightarrow \text{comp}_{\vec{u}}\vec{v} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|} = \dfrac{12}{3\sqrt{6}} = \dfrac{4}{\sqrt{6}}$

$\dfrac{\vec{u}}{|\vec{u}|} = \dfrac{1}{3\sqrt{6}}\langle 7,1,-2\rangle$

$$\Rightarrow \text{proj}_{\vec{u}}\vec{v} = \left(\frac{4}{\sqrt{6}}\right)\left(\frac{1}{3\sqrt{6}}\langle 7,1,-2\rangle\right) = \frac{4}{3\cdot 6}\langle 7,1,-2\rangle = \frac{2}{9}\langle 7,1,-2\rangle$$

$$= \left\langle \frac{14}{9}, \frac{2}{9}, \frac{-4}{9}\right\rangle$$

<u>Note</u>: $\vec{v} - \text{proj}_{\vec{u}}\vec{v}$ is orthogonal to $\vec{u}$.

As in the example:

$$\vec{v} - \text{proj}_{\vec{u}}\vec{v} = \langle 3, -5, 2 \rangle - \left\langle \frac{14}{9}, \frac{2}{9}, \frac{-4}{9} \right\rangle = \left\langle \frac{13}{9}, \frac{-47}{9}, \frac{22}{9} \right\rangle$$

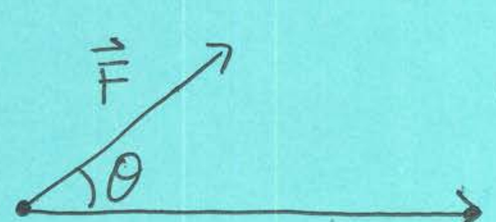
$$= \vec{w}$$

$$\vec{w} \cdot \vec{u} = \left(\frac{13}{9}\right)(7) + \left(-\frac{47}{9}\right)(1) + \left(\frac{22}{9}\right)(-2)$$

$$= \frac{91}{9} - \frac{47}{9} - \frac{44}{9} = 0.$$

<u>Application</u>: Work

Suppose a force $\vec{F}$ acts on an object that moves along a displacement vector $\vec{D}$ from p to q.



The work done on the object by $\vec{F}$ is the component of the force in the direction of the motion times the displacement distance.

In symbols:

$$\text{Work} = W = (|\vec{F}|\cos\theta)\,|\vec{D}| = \vec{F} \cdot \vec{D}$$